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# SIMPLIFIED EQUATIONS OF MOTION FOR THE RADIAL–AXIAL VIBRATIONS OF FLUID FILLED PIPES

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The equations of motion for straight fluid filled pipes are greatly simplified. It is found, for frequencies below a third of the ring frequency, that the radial–axial waves in cylinders are as if the circumferential motion were inextensional. This is the fundamental assumption for the analysis. The derivation is also based on the assumption of long axial wavelengths, resulting in the axial inertia of the fluid and the axial flexural stiffness of the pipe wall being negligible. The formulation is restricted to frequencies well below the cut-on of higher order fluid modes. For such frequencies, the compressibility of the fluid is neglected and the internal fluid loading, on the pipe, is approximated as an increase in the radial inertia. Upon this basis, the equations of motion, for each circumferential mode, are similar to those for a Timoshenko beam on a Winkler foundation. Numerical experiments are made, comparing the approximate theory with results from calculations from the Helmholtz equation for the fluid and accurate thin-walled cylinder theory. Criteria for the application of the simplified theory are formulated.



## INTRODUCTION

Pipes for conveying liquids are frequent components in industries and in vehicles. Vibrations in pipes are excited by mechanical forces, by pumps and by turbulent pressure fluctuations occurring at geometric irregularities such as valves, orifices and junctions. Once excited, the vibro-acoustical waves may transmit quite easily through the system, often ending up where they can cause noise problems or failure due to fatigue.

The waves in a fluid filled pipe, i.e., the solutions to the coupled equations of motion, were originally investigated by Fuller and Fahy [1]. The analysis was later extended by Fuller to cope with forced response in infinite pipes that are excited by point forces [2] and by monopole sources in the fluid [3]. Also, Fuller considered sound radiation [4] and Pavic [5] derived expressions for the energy flow. Feng [6] introduced a thin elastic layer in between pipe and fluid, modelling air bubbles and pipe imperfections, thereby achieving a better agreement between measurements and calculations.

The approach used in references [1–6], namely solving the characteristic equation, once the sound pressure has been expressed as a linear function of the radial displacement, results in a non-linear eigenvalue problem, which is non-trivial to handle. To overcome this, Finnveden [7] developed a FE technique which is numerically stable and very efficient. The solutions of the equations of motion achieved with the FE technique are then used as base functions in a spectral (i.e., frequency dependent) FE formulation for arbitrarily

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long pipes. The routines presented in reference [7] are used here to assess the accuracy of the developed approximate theory.

The behaviour of a pipe is quite different for frequencies above and below the ring-frequency, this frequency occurring when the in-plane extensional wavelength in the pipe is equal to its circumference. However, pipes for conveying liquids most often have diameters in the range 1 cm–1 m, so, for steel pipes the ring-frequency is in the range 160 kHz–1·6 kHz. Consequently, many noise and vibration problems are related to frequencies well below the ring-frequency, and only such frequencies are considered here.

The first step in an analysis of pipes is to decompose the motion's circumferential dependence in a Fourier series. In this series, the first, $n = 0$, term has no circumferential dependence; the $n = 1$ term has one wavelength around the circumference; the $n = 2$ term has two wavelengths; and so on. At lower non-dimensional frequencies, there are four types of waves that can propagate in fluid filled pipes: (1) torsional, $n = 0$, waves in the pipe wall which are not coupled to the fluid motion; (2) beam-type, $n = 1$, bending waves with the fluid mass added to that of the pipe; (3) coupled axisymmetric, $n = 0$, dilatational waves which are predominately as: (a) longitudinal waves in the pipe wall with some radial motion because of "Poisson coupling" or (b) plane acoustic waves forcing some radial motion of the pipe wall.

At somewhat higher frequencies, dependent on the wall thickness, higher order radial–axial ($n = 2$, "ovaling", $n = 3$, "teddy bear", etc.) waves can propagate. In addition, at even higher frequencies, there is the axisymmetric flexural wave, cut on at the ring-frequency, and non-planar predominantly acoustic waves as well as non-planar longitudinal and torsional waves. For commonly used engineering materials, the first of these waves that can propagate is the first higher order acoustic wave, the $n = 1$ mode. For a water filled steel pipe, this mode is cut on approximately at half the ring-frequency; whereas for an air filled steel pipe, it is cut on around an eighth of the ring-frequency.

To predict transmission of vibrations in pipe systems, and the effect of these vibrations, accurate mathematical models are needed and further development is called for. To understand the implications of different designs, however, it would be beneficial to have simple theory, that may illuminate the important physical properties: hence, a theory that is not too elaborate. For obvious reasons, the low frequency vibrations of an empty pipe are described with the cylinder treated as a beam. Such a description is also accurate for a fluid filled pipe if the fluid–structure coupling effects are accounted for [8, 9]. Such equivalent beam models are applied for low frequency predictions of vibration transmission in pipe structures by using the transmission matrix method [10–12] and the finite element method [13].

For fluid filled pipes the simplified models for axisymmetric motion are accurate at low frequencies, whereas the Euler beam model, for the $n = 1$ radial–axial mode, is only fairly accurate [9]. For higher order shell modes, no similar simplified theory exists. Developing such theory is the objective of the present work. It appears that it has not previously been recognized that equivalent beam models can describe pipe vibrations also at frequencies where higher order shell modes, with several circumferential wavelengths, are cut on.

For frequencies well below the ring-frequency the restraint against cross-sectional "breathing" is very large. In thin walled beam theory [14] it is often assumed that the in-plane strain in the cross-section is zero. For a ring, this corresponds to the inextensional ring theory of Rayleigh [15]. For cylinders, this assumption is supported by the low frequency results in reference [16, Table 2.11], where it is seen that the mode shapes are close to those achieved by inextensional theory.

In accordance with this, the theory developed is based on the assumption of zero circumferential in-plane strain. After applying a Fourier decomposition of the displacement's angular dependence, the tangential in-plane displacement is then eliminated from the problem. When it is also assumed the axial wavelengths are long compared to the cylinder radius, the fluid loading can be expressed as a linear function of the radial displacement. Also, for long wavelengths, the axial bending of the cylinder wall could be disregarded. Upon this basis, the equations of motion for fluid filled pipes are equivalent to those for a Timoshenko beam on a Winkler foundation; this "spring foundation" describes the circumferential flexural stiffness of the pipe.

In this work simple models are developed, showing that the propagating radial–axial waves ($n = 1, 2, \ldots$) in fluid filled pipes can be described with equivalent beam theory. Criteria for applying the simplified theory are derived and verified by numerical experiments. By using this theory, vital characteristics of an acoustic problem such as modal density and input mobility for mechanical point forces and fluid monopole sources can be defined by closed form expressions, as will be reported at a later stage [17].

## 2. EQUIVALENT BEAM MODEL FOR THE RADIAL-AXIAL MOTION OF FLUID FILLED PIPES

### 2.1. CYLINDER EQUATIONS

The motion of thin walled cylinders is investigated by using a Fourier decomposition of the circumferential dependence of the displacements and assuming plane stress in the pipe-wall cross-section according to the Kirchhoff hypothesis. The displacements, as shown in Figure 1, are

$$u_x = (u + z\theta_1)\cos(n\phi), \qquad u_\phi = (v + z\theta_2)\sin(n\phi), \qquad u_z = w\cos(n\phi), \tag{1}$$

where, by using the plane stress condition [16, equation 1.39], one has

$$\theta_1 = -\partial w/\partial x, \qquad \theta_2 = (v + nw)/R \tag{2}$$

Accordingly, the strains are [18]

$$e_x = (\varepsilon_x + z\kappa_1)\cos(n\phi), \qquad e_\phi = (\varepsilon_\phi + z\kappa_2)\cos(n\phi), \qquad \gamma_{x\phi} = (\gamma + z\tau)\sin(n\phi), \tag{3}$$

where

$$\varepsilon_x = \partial u/\partial x, \qquad \varepsilon_\phi = (nv + w)/R, \qquad \gamma = -(nu/R) + \partial v/\partial x,$$

$$\kappa_1 = -\partial^2 w/\partial x^2, \qquad \kappa_2 = (nv + n^2 w)/R^2, \qquad \tau = (2/R)(\partial v/\partial x + n\partial w/\partial x). \tag{4}$$

Stationary time dependence of the form $e^{-i\omega t}$ is assumed. The equations of motions are, as in reference [7], derived by using a modified version of Hamilton's principle applicable also for non-conservative motion. Dissipative losses, possibly frequency dependant, are

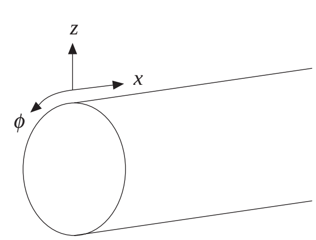


Figure 1. The cylinder co-ordinate system.

assumed proportional to either inertia or stiffness. That is, the equations below apply equally when

$$E = E_0(1 - i\eta_e), \qquad G = G_0(1 - i\eta_s), \qquad \rho = \rho_0(1 + i\eta_v), \tag{5}$$

where $E$ is Young's modulus, $G$ is the shear modulus, $\rho$ is the density, and $\eta_e$, $\eta_s$, and $\eta_v$ are loss factors.

By using the modified Hamilton's principle, the quadratic forms in the displacements, expressing the potential and kinetic energy densities, are replaced with symmetric bi-linear forms in the displacements and the displacements of an adjoint, negatively damped system. With no losses, the resulting functional is equal to the standard Lagrangian. If the energy densities, corresponding to the displacements (1) and the strains (3), are integrated over the cross-section as done by Arnold and Warburton [18], the functional $L_{cyl}$ governing the motion of a thin walled cylinder is

$$\begin{aligned} L_{cyl} = E' T_c R\Lambda_n \int & [(\varepsilon_x^a \varepsilon_x + \varepsilon_\phi^a \varepsilon_\phi + \nu(\varepsilon_x^a \varepsilon_\phi + \varepsilon_\phi^a \varepsilon_x) + g\gamma^a\gamma) \\ & + (T_c^2/12)(\kappa_1^a \kappa_1 + \kappa_2^a \kappa_2 + \nu(\kappa_1^a \kappa_2 + \kappa_2^a \kappa_1) + g\tau^a\tau) \\ & - \omega^2/c_L^2 (u^a u + v^a v + w^a w)] \, dx, \end{aligned} \tag{6}$$

where

$$E' = E/(1 - \nu^2) \qquad g = G/E' = (1 - \nu)/2, \qquad c_L^2 = \rho/E', \tag{7}$$

$$\Lambda_0 = 2\pi, \qquad \Lambda_n = \pi, \quad n \geqslant 1 \tag{8}$$

$R$ is the cylinder radius, $T_c$ is the shell thickness and $\nu$ is Poisson ratio. The upper index $a$ denotes the complex conjugate of the corresponding strain in the adjoint system; with no losses this would be the complex conjugate of the strain.

Thus, requiring $L_{cyl}$ to be stationary is equivalent to requiring the displacements to be solutions to the equations of motion, (see, e.g. [16, equation 2.9b]). When these equations are written in uni-dimensional form [16], it is seen that the free vibrations of a pipe is a function of solely the trigonometric order, $n$, the Poisson ratio $\nu$, the parameter $\beta$ and non-dimensional frequency $\Omega$:

$$\beta = T_c^2/12R^2, \qquad \Omega = \omega R/c_L. \tag{9}$$

In equation (6), the first term is the potential energy from in-plane membrane strain, the second term is the potential energy from bending of the shell wall and the last term is the kinetic energy. Within the expressions for the potential energies, the first terms are due to axial strain, the second terms are due to circumferential strain whereas the fourth terms are from in plane shear and from twist of the shell wall, respectively. To investigate the relative importance of these contributions, the dispersion relations for a cylinder without any losses and with $T_c = R/30$, $\nu = 0{\cdot}3$ and $n = 2$ and $n = 4$ were solved by using the routines in reference [7]. For $\Omega < 1$ there is, for each $n \geqslant 2$, only one propagating wave. By using the calculated mode shapes for these waves, the potential energy for each of the terms was calculated. The results were then divided by the total potential energy to produce the relative contribution of each term.

In Figures 2 and 3 these relative contributions to the potential energy are shown. (Note that, as the Poisson coupling terms might be positive or negative, the sum of the relative contributions of the other six terms may be both larger and smaller than 1.) At cut-on of a wave, the most important term is the circumferential bending of the shell wall. At

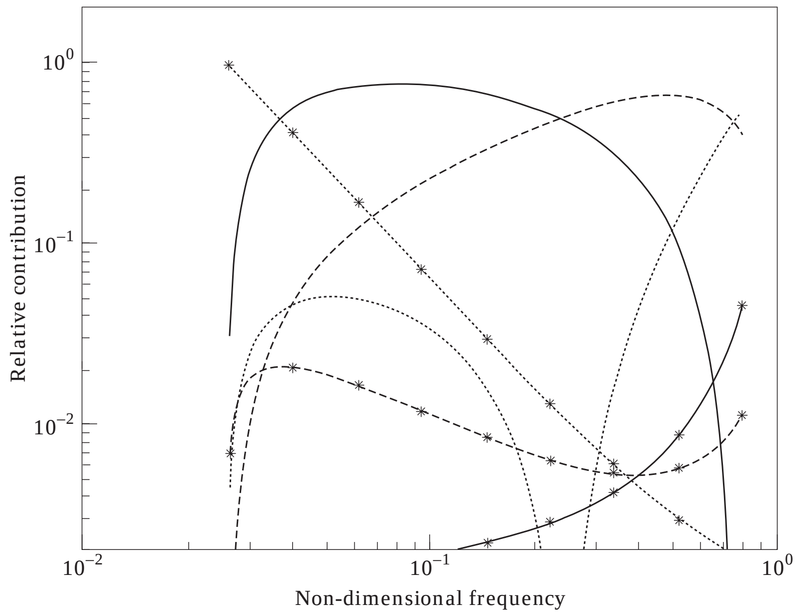


Figure 2. Relative contributions to the total potential energy in the $n = 2$, propagating wave. ——, Axial in-plane; - - - -, in-plane shear; · · · · , circumferential in-plane; —*—*–, axial bending; - -*- - -*- -, bending twist; · · ·*· · ·, circumferential bending.

somewhat higher frequencies, the in-plane axial term, that is, the cross-sectional bending, dominates. At even higher frequencies the in-plane shear is important. The axial bending of the shell wall is negligible but for frequencies close to the ring-frequency. The twist gives an almost constant contribution which for $n = 2$ is small and for $n = 4$ is somewhat larger;

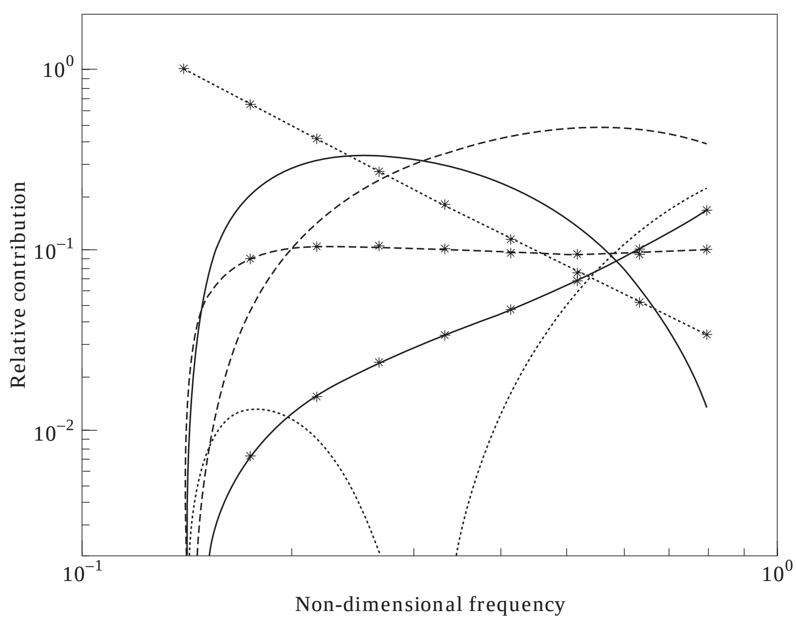


Figure 3. As Figure 2, but $n = 4$.

this term is significant only for large $n$ and $\beta$. Most important, the in-plane circumferential strain energy is small for all frequencies below half the ring-frequency.

The modelling of the $n = 0$, axisymmetric motion of fluid filled pipes has been considered in, e.g., references [9] and [19]. Here the analysis is restricted to $n \geqslant 1$. Now, upon assuming that the potential energy from circumferential in-plane strain is negligible,

$$\varepsilon_\phi = (nv + w)/R = 0 \qquad v = -w/n. \tag{10}$$

At lower frequencies, this assumption is in reference [16, Table 2.11] seen to be in good accordance with results achieved by using the Flügge theory. Using (10) and substituting

$$u = R\theta/n^2, \tag{11}$$

yields the strains in the cylinder, equations (4), as

$$\varepsilon_x = \frac{R}{n^2}\frac{\partial\theta}{\partial x}, \qquad \varepsilon_\phi = 0, \qquad \gamma = -\frac{1}{n}\left(\theta + \frac{\partial w}{\partial x}\right),$$

$$\kappa_1 = -\frac{\partial^2 w}{\partial x^2}, \qquad \kappa_2 = \frac{n^2-1}{R^2}\,w, \qquad \tau = \frac{2}{R}\left(n - \frac{1}{n}\right)\frac{\partial w}{\partial x}. \tag{12}$$

The potential energy from bending of the shell wall is given by the strains and $\kappa_1$, $\kappa_2$ and $\tau$. It is seen to be a functional of only the radial displacement, $w$, so the terms may be compared. For $n = 1$, and as long as the axial wavelength is not shorter than the radius, the membrane theory, with entire neglect of the potential energy due to bending of the pipe wall, gives excellent results [16, Table 2.7]. For $n \geqslant 2$ and wavelengths not shorter than $6*R$, it is seen in equation (12), that the contributions to the potential energy from axial bending, and from twist, are less than that from circumferential bending. Consequently, at lower frequencies, the errors introduced when neglecting terms proportional to $\kappa_1$ and $\tau$ are believed to be small. Upon assuming this, the functional $L_{cyl}$, equation (6), is

$$L_{cyl} = \int\Bigg[E'I_n\frac{\partial\theta^a}{\partial x}\frac{\partial\theta}{\partial x} + GAK_n\left(\theta^a + \frac{\partial w^a}{\partial x}\right)\left(\theta + \frac{\partial w}{\partial x}\right)$$
$$+\beta E'A/2\left(\frac{n^2-1}{R}\right)^2 w^a w - \rho\omega^2 A/2\left(1 + \frac{1}{n^2}\right)w^a w - \rho\omega^2 I_n\,\theta^a\theta\Bigg]\,\mathrm{d}x, \tag{13}$$

where

$$A = 2\pi T_c\,R, \qquad K_n = 1/(2n^2), \qquad I_n = I_y/n^4, \qquad I_y = \pi T_c\,R^3. \tag{14}$$

This functional describes an equivalent Timoshenko beam on a Winkler foundation. The cylinder has an equivalent area moment of inertia $I_n$ and shear coefficient $K_n$. The equivalent mass is $M_e$, while the spring constant for the Winkler foundation is $K_w$:

$$M_e = \rho A/2\left(1 + \frac{1}{n^2}\right), \qquad K_w = \beta E'A/2\left(\frac{n^2-1}{R}\right)^2. \tag{15}$$

For $n = 1$, the functional describes the pipe modelled with standard Timoshenko beam theory using a shear coefficient $K = 1/2$. This value is close to that given in reference [20] for a thin walled cylinder: $K = (2 + 2\nu)/(4 + 3\nu)$.

Now, by varying the adjoint displacements, the equations of motion are found to be

$$E'I_n\ \partial^2\theta/\partial x^2 = GAK_n\,(\theta + \partial w/\partial x) - \rho\omega^2 I_n\,\theta, \tag{16a}$$

$$GAK_n\,(\partial/\partial x)\,(\theta + \partial w/\partial x) = (K_w - \omega^2 M_e)w, \tag{16b}$$

If also the potential energy from twist is included in $L_{cyl}$, equation (16b) is replaced by

$$GAK_n\left[\frac{\partial}{\partial x}\left(\theta + \frac{\partial w}{\partial x}\right) + C_n\frac{\partial^2 w}{\partial x^2}\right] = (K_w - \omega^2 M_e)w, \tag{16c}$$

where

$$C_n = 2\beta(n - 1/n)^2/K_n. \tag{17}$$

This system of equations may be expanded to form a set of four first order ordinary differential equations. This system has constant coefficients, so the solutions are of the form $e^{ik_n x}$. After assuming this, the resulting linear eigenvalue problem is solved by standard methods. Alternatively, solving the characteristic equation, yields the propagating wavenumbers given by the non-parenthesized signs in

$$k_n R = {}^{+}_{(-)}[H^{+}_{(-)}[H^2 + M(Q - \Omega^2)]^{1/2}]^{1/2}, \tag{18}$$

where

$$H = (M + \Omega^2 - QC_n/(1 + C_n))/2,$$

$$M = (\omega^2 M_e - K_w)R^2/(GAK_n\,(1 + C_n)),$$

$$Q = (GAK_n\,R^2)/(E'I_n) = n^2 G/E'. \tag{19}$$

If the restraints against twist of the shell wall are not considered $C_n = 0$.

In Figures 4–7 are shown the propagating wavenumbers calculated with this approximate theory and with the more accurate Arnold and Warburton theory by using the routines in reference [7]. For $n = 1$, the results are in good agreement up to the cut-on of the torsional wave at approximately $\Omega = 0{\cdot}7$. Surprisingly, perhaps, this cut-on is, described with the inextensional theory. For $n \geqslant 2$ and for frequencies $\Omega < 0{\cdot}1$, the results are also in good agreement. For low order $n$, or for very thin-walled pipes, the agreement is considerable at even higher frequencies. The significance of the twist increases for large $n$ and $\beta$. It is concluded, for all waves having cut-on at frequencies below $\Omega = 0{\cdot}1$, the approximate theory is good up to almost $\Omega = 0{\cdot}5$ with an accuracy sufficient in most noise and vibration control problems.

The resonance frequencies for the radial–axial modes are in reference [16, Tables 2.14 and 2.15] seen to be almost independent of the Poisson ratio $\nu$. Therefore, as the free vibrations of cylinders are a function of only $\Omega$, $\nu$, $\beta$, and $n$, the results in Figures 4–7 cover the principal characteristics of the dispersion relations for propagating waves in thin walled cylinders at frequencies below half the ring-frequency.

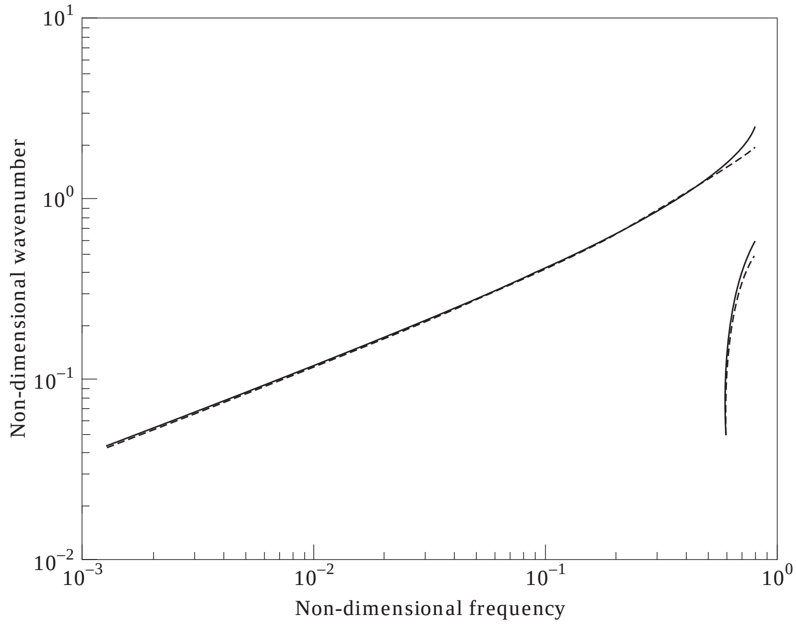


Figure 4. Wavenumbers times radius for an empty pipe, $n = 1$, $T_c = R/180$ and $T_c = R/60$ and $T_c = R/20$. ——, Arnold and Warburton theory; - - - -, Timoshenko beam theory, equations (18) and (19).

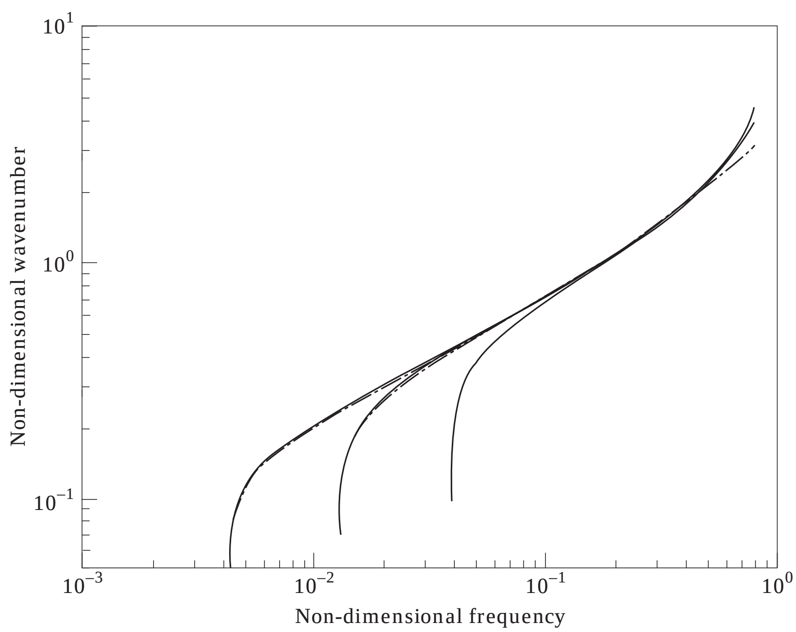


Figure 5. Wavenumbers times radius for an empty pipe, $n = 2$, $T_c = R/180$, to the left, $T_c = R/60$, middle, and $T_c = R/20$, to the right. ——, Arnold and Warburton theory; - - - -, equivalent Timoshenko beam theory, equations (18) and (19) with $C_n = 0$; · · · · , equivalent Timoshenko beam theory plus bending twist, equations (18) and (19).

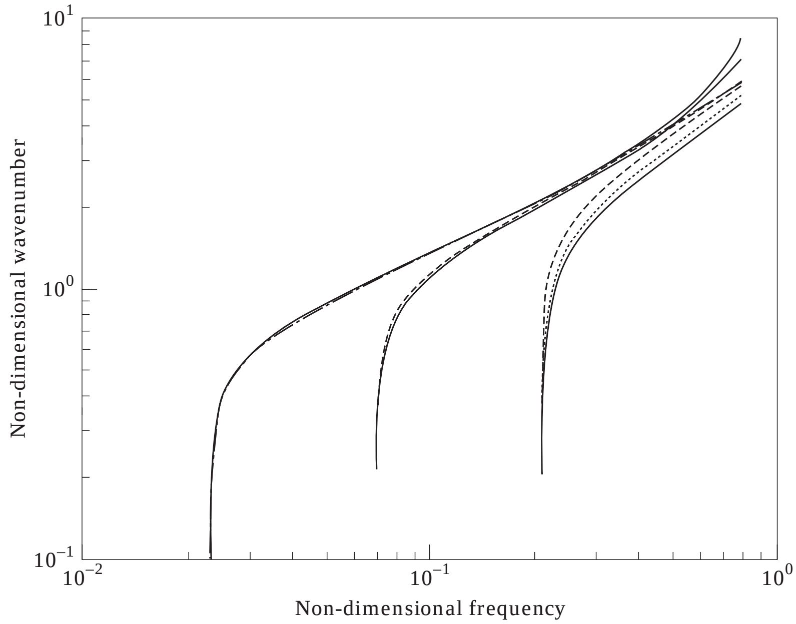


Figure 6. As Figure 5, but $n = 4$.

### 2.2. FLUID FILLED PIPE

The functional governing a fluid filled pipe vibration is in reference [7] found to be

$$L = L_{cyl} - B_{fc} - L_f, \tag{20}$$

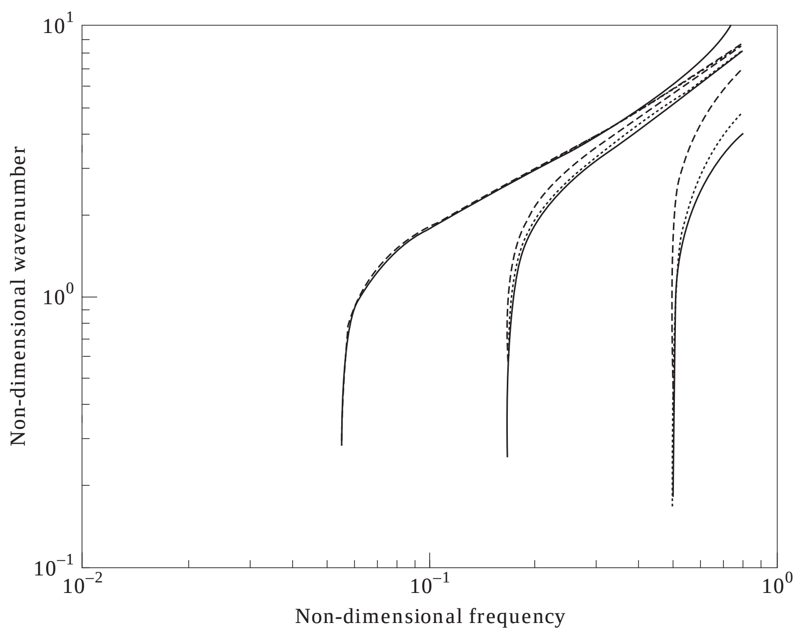


Figure 7. As Figure 5, but $n = 6$.

where

$$L_f = -\Lambda_n \frac{\rho_f}{1+\mathrm{i}\eta_v} \int\left[\left(\frac{\partial\psi^a}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\psi^a}{\partial r}\frac{\partial\psi}{\partial r} + \frac{n^2}{r^2}\psi^a\psi\right) - \frac{\omega^2(1+\mathrm{i}\eta_v)}{c_f^2(1-\mathrm{i}\eta_e)}\psi^a\psi\right] r\,\mathrm{d}r\,\mathrm{d}x, \quad (21)$$

$$B_{fc} = \Lambda_n \rho_f \omega R \int [\psi^a w + \psi w^a]\,\mathrm{d}x, \quad (22)$$

and where

$$p(x, r, \phi) = \rho_f \omega \cos n\phi \psi(x, r). \quad (23)$$

$p$ is the sound pressure in the fluid, $c_f$ is the sound speed, $\rho_f$ is the density and $\eta_v$ and $\eta_e$ are loss factors.

The solutions of the coupled equations of motion are [1]

$$\psi = F(x)\mathrm{J}_n(k_f r)\cos n\phi, \qquad k_f R = \sqrt{\Omega^2(c_L/c_f)^2 - (k_n R)^2}, \quad (24)$$

where the wavenumber $k_n$ and the function $F$ are found from the dispersion relations. For both real and complex $k_f$, at lower frequencies this simplifies to

$$\psi = f(x) r^n \cos n\phi. \quad (25)$$

Upon assuming this, $B_{fc}$ and $L_f$ are

$$L_f = -\frac{\Lambda_n \rho_f}{1+\mathrm{i}\eta_v} R^{2n} \int\left[\frac{R^2}{2n+2}\frac{\partial f^a}{\partial x}\frac{\partial f}{\partial x} + n f^a f - \frac{\omega^2(1+\mathrm{i}\eta_v)}{c_f^2(1-\mathrm{i}\eta_e)}\frac{R^2}{2n+2} f^a f\right]\mathrm{d}x, \quad (26)$$

$$B_{fc} = \Lambda_n \omega \rho_f R^{n+1}[f^a w + f w^a]. \quad (27)$$

The approximation (25) is valid only at low frequencies, when $k_f R \ll 1$. Also, at low frequencies, for propagating radial–axial modes, $k_f \approx \mathrm{i}k$. Consequently, at lower frequencies, the first and third terms in $L_f$ are neglected. That is, only the cross-sectional inertia is included, while the axial inertia and the compressibility of the fluid are neglected. Upon this, considering the functional $L$, equation (20), and varying $f^a$ results in that $f$ is determined by

$$f = \omega(1+\mathrm{i}\eta_v) w / n R^{n-1}, \quad (28)$$

and the equations of motion for the fluid filled pipe are as in equations (16) with $M_e$ given by

$$M_e = \rho A/2\left(1 + \frac{1}{n^2} + \frac{2\mu}{n}\right), \quad (29)$$

where $\mu$ is the ratio of the masses, per unit length, in the cylinder and in the fluid

$$\mu = R\rho_f / 2T_c \rho. \quad (30)$$

In equations (29) and (30) the influence of the viscous losses in the fluid is not explicitly expressed while it is included if $\rho_f$ is replaced by $\rho_f(1+\mathrm{i}\eta_v)$.

TABLE 1

*Material parameters*

| Material | Poisson ratio, $\nu$ | Density, $\rho$ (kg/m$^3$) | Free wave speed, $\sqrt{E/\rho}$; $c_f$; (m/s) |
|---|---|---|---|
| Steel | 0·3 | 7800 | 5196 |
| Water | — | 1000 | 1500 |

### 2.3. EULER BEAM APPROXIMATION

At very low frequencies, the cross-sectional shear in a beam is small and the Euler beam theory is accurate. Applying this theory yields the wavenumber for a propagating wave in a liquid filled pipe as

$$k_n = [(\omega^2 M_e - K_w)/E'I_n]^{1/4}. \tag{31}$$

Following the discussion in reference [21, section 2.3], this approximation should be valid as long as

$$EI_n k_n^2 / GAK_n < C, \tag{32}$$

where $C$ is a constant which, by Cremer and Heckl [21], for a beam with rectangular cross-section, was set to $C = E/10G$. Upon using this value and the Euler beam approximation of the wavenumbers (31), while neglecting the Winkler foundation stiffness, which does not influence the wavenumber well above cut-on, the criterion (32) becomes

$$\Omega < 1/10\sqrt{1 + 1/n^2 + 2\mu/n}. \tag{33}$$

For $n = 1$, and small values of $\mu$, this formula agrees approximately with the value given by de Jong [10]: $\Omega \leqslant 0{\cdot}05$. For a thin walled pipe and a dense fluid, the limiting frequency is lower than this. Surprisingly, perhaps equation (33) implies that the frequency range for which the equivalent Euler beam theory applies increases somewhat for larger $n$.

Upon employing the Euler beam approximation, it is seen in equation (31) that the cut-on frequencies are given by

$$\omega^2_{cut\text{-}on} M_e = K_w, \qquad \Omega^2_{cut\text{-}on} = \frac{\beta(n^2 - 1)^2}{1 + 1/n^2 + \mu/n}. \tag{34}$$

Apart from terms proportional to $\beta^2$, this is the result obtained by Pavic [5], using the Flügge theory. Thus it is concluded that for frequencies around the cut-on frequencies, the Euler beam approximation is accurate.

## 3. NUMERICAL EXPERIMENTS

The motion of a fluid filled pipe is described by the non-dimensional numbers $\Omega$, $n$, $\nu$, $\beta$, $c_f/c_L$ and $\rho_f/\rho$. The results from the theory presented here for various combinations of these numbers are compared to the results found by using the routines in reference [7], based on the Helmholtz equation for the fluid and accurate thin walled cylinder theory. Thus it is possible to verify the approximate equivalent beam theory and to find limits for its application. Unless explicitly stated, the experiments were made for water filled steel pipes the material data being given in Table 1.

### 3.1. WATER FILLED STEEL PIPES

The wavenumbers for the propagating radial–axial modes in water filled steel pipes were calculated by using the routines in reference [7], the equivalent Timoshenko beam theory, including twist, equations (18), (19) and (29), and the equivalent Euler beam theory, equation (31). The non-dimensional wavenumbers, $k_n R$, are shown, for $n = 1$, 2, 4 and 6, in Figures 8–10 for $T_c = R/20$, $T_c = R/60$ and $T_c = R/180$, respectively. The limiting frequency for the application of the Euler beam theory is, as anticipated by equation (33), determined by the $n = 1$ mode. It is for the $T_c = R/20$ pipe at $\Omega \approx 0{\cdot}04$ and for the $T_c = R/180$ pipe at $\Omega \approx 0{\cdot}015$. The limiting frequency for applying the Timoshenko beam theory is also determined by the $n = 1$ mode. For the thin walled pipe it is $\Omega \approx 0{\cdot}15$, whereas it is perhaps twice this frequency for the thick walled pipe. Notably, the cut-on frequencies are accurately estimated by the Euler beam theory.

### 3.2. INFLUENCE OF FLUID COMPRESSIBILITY

In the derivation of the equivalent beam theories for fluid filled pipes, the compressibility of the fluid is neglected. Hence the wavenumbers calculated by equations (18) and (31) are independent of the fluid sound speed. To investigate this, the wavenumbers were calculated, for the $n = 1$ wave, for a pipe with wall thickness $T_c = R/60$ and a fluid with density as that of water but with sound speeds $c_f = 100$, 400, 1600 or 6400 m/s. The results are shown in Figure 11. For the compliant fluids, it is seen that the limiting frequency for the beam theory is of the order of half the cut-on frequency for the first fluid mode. For a rigid pipe, this limiting frequency is given by

$$\Omega = 1{\cdot}84/2^* c_f / c_L. \tag{35}$$

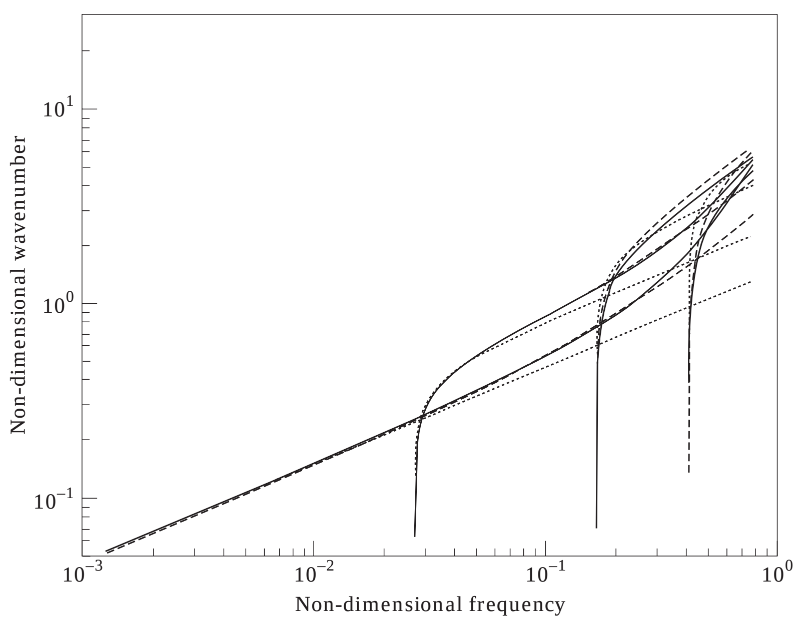


Figure 8. Wavenumbers times radius for a water filled steel pipe, $T_c = R/20$, $n = 1$, 2, 4 and 6. ——, Arnold and Warburton theory; - - - -, equivalent Timoshenko beam theory plus bending twist, equations (18), (19) and (29); · · · · , equivalent Euler beam theory, equation (31).

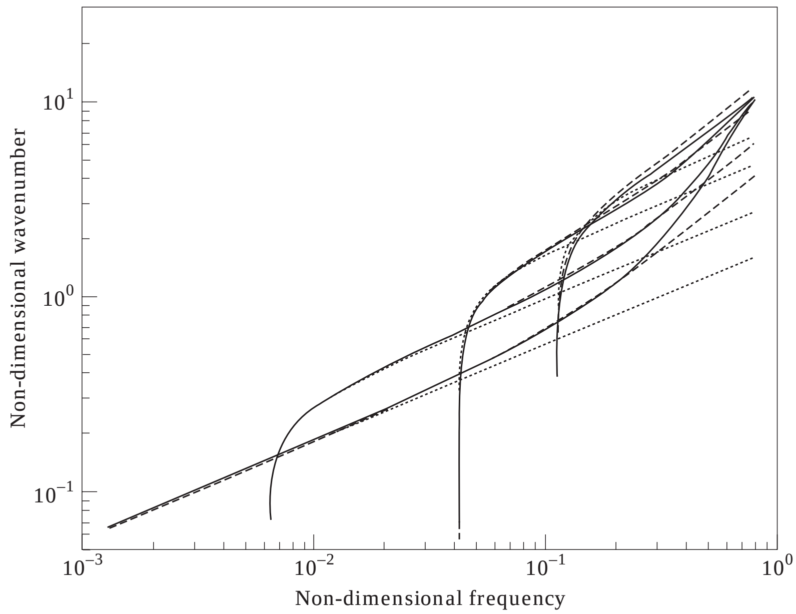


Figure 9. As Figure 8, but $T_c = R/60$.

### 3.3. LONG WAVELENGTH ASSUMPTION

Besides neglecting the fluid compressibility, the axial inertia of the fluid is neglected and the cross-sectional mode shapes for the fluid are approximated as in equation (25). For the propagating mode, the radial wavenumber is approximately $k_f \approx ik$, where $k$ is the axial wavenumber. This means that when the non-dimensional wavenumber, $kR$, is of the order of 1, the fluid's motion is not accurately modelled by the trial function (25).

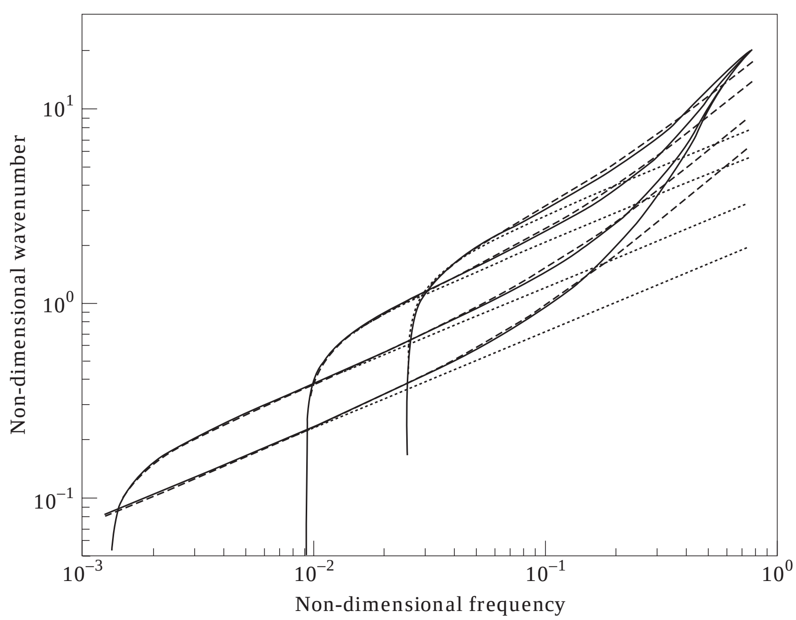


Figure 10. As Figure 8, but $T_c = R/180$.

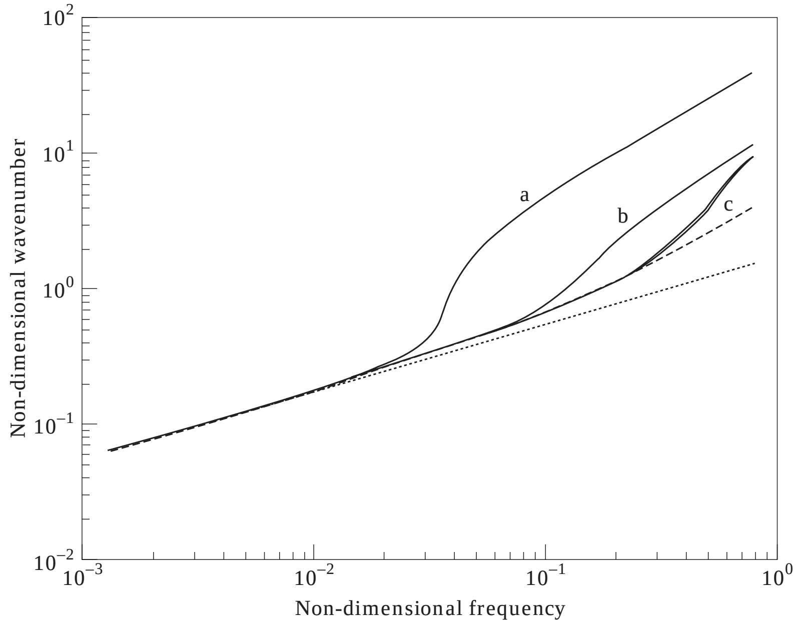


Figure 11. Wavenumbers times radius for a fluid filled steel pipe, $T_c = R/60$, $n = 1$. a, $c_f = 100$ m/s; b, $c_f = 400$ m/s; c, $c_f = 1600$ m/s and $c_f = 6400$ m/s; ——, Arnold and Warburton theory; - - - -, equivalent Timoshenko beam theory plus bending twist, equations (18), (19) and (29); · · · · , equivalent Euler beam theory, equation (31).

The approximation of the cylinder to an equivalent beam is also based on the assumption of long axial wavelengths. In Figure 12 are shown the relative contributions to the total potential energy in the shell wall for the $n = 1$ propagating wave in a water filled steel pipe with thickness $T_c = R/180$. At $\Omega \approx 0{\cdot}15$, the circumferential in-plane strain becomes the dominating term, so the assumption of inextensional circumferential motion is not true. At this frequency, it is seen in Figure 10 that the non-dimensional wavenumber is slightly more than 1. Comparison with Figures 8 and 9 shows that also for pipes with thicker walls the description of the water filled pipe as a beam fails when the non-dimensional wavenumber is slightly more than 1. The Euler beam theory predicts somewhat smaller wave numbers than the Timoshenko theory; thus the criterion is expressed more conveniently—when the non-dimensional wavenumber predicted by the Euler beam theory is of the order of 1. Thus, for frequencies

$$\Omega < 1/\sqrt{2 + 2\mu}, \tag{36}$$

the propagating waves in fluid filled steel pipes are accurately modelled by equivalent beam theory; whereas, for higher frequencies, the assumed cross-sectional fluid motion is not correct, nor is the neglect of the axial fluid inertia, nor the neglect of axial flexural stiffness of the pipe wall. Most importantly, for frequencies above this, the assumption of in-extensional theory is not valid, so the entire approach fails.

### 3.4. INFLUENCE OF FLUID DENSITY

To verify the conclusions drawn on the applicability of the equivalent beam theory, different fluid densities were considered. Calculations are made for a thin walled pipe $T_c = R/180$ filled with a fluid having sound velocity $c_f = 1500$ m/s and densities 6400, 1600, 400 and 100 kg/m$^3$. In Figure 13 are shown the wavenumbers for the propagating $n = 1$ beam mode, where the uppermost curves correspond to the highest density. It is seen that the application of the beam theory is restricted to frequencies below $\Omega \approx 0{\cdot}25$, this frequency corresponding to half the cut-on frequency for the first fluid mode. Besides this, the Timoshenko beam theory is accurate for frequencies below those predicted by criterion (36), and the Euler beam theory is accurate for frequencies of the order of a factor of 10 below this, as predicted by equation (33).

### 3.5. ACCURACY AT LOW FREQUENCIES

Finally, the accuracy of the beam approximation at very low frequencies was investigated. All the neglected terms in the original functional $L$, equation (20) with equation (6) used for $L_{cyl}$, vanish as the frequency tends to zero. Hence, at low frequencies the errors introduced are due to the assumed restraints on the motion, imposed in equation (10). Now, when using a variational principle, restraints on motion result in an overestimation of the stiffness and the inertia terms. In Figures 4 and 5 it is seen that at low frequencies the wavenumbers are underestimated; hence, the stiffness of the pipe is overestimated by the beam theories.

Pavic used an Euler beam approximation to describe the $n = 1$ bending vibrations of fluid filled pipes [9]. He found that the errors depend not only on frequency but also on the Poisson ratio. Pinnington and Briscoe [19] found that the axial stiffness of the $n = 0$ longitudinal mode is that of a rod, $EA$, at low frequencies, while it is that of a plate, $EA/(1 - \nu^2)$, at higher frequencies. Now, it is tempting to use the rod value of Young's modules instead of the plate value in the cross-sectional bending term, to correct the beam approximation at lower frequencies. Numerical experiments reveal that this improves the calculated wavenumbers, compared with those found by the Arnold and Warburton theory [7]. Support for this idea is also given in Figure 12, where it is seen that at low frequencies; (1) when neglecting the Poisson coupling terms, the total potential energy is overestimated and (2) the neglected in-plane circumferential stiffness is not vanishingly small. This

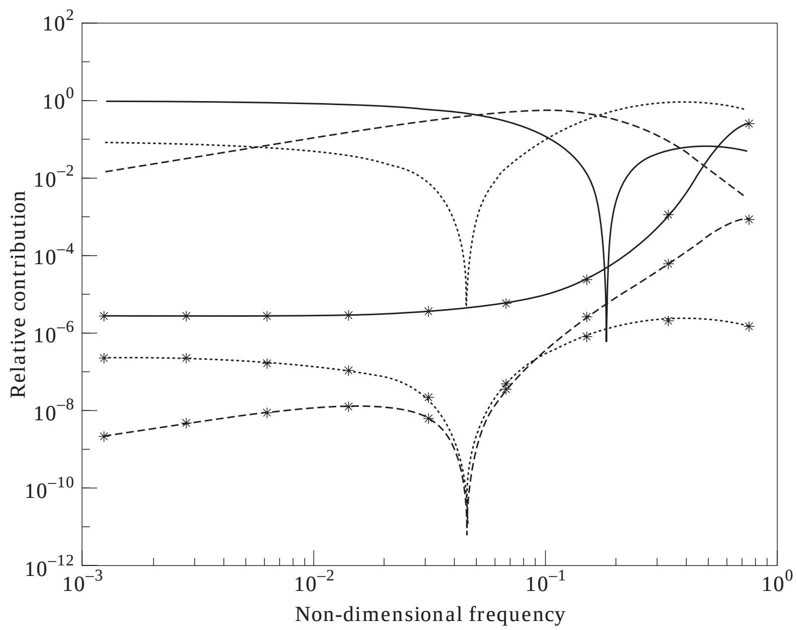


Figure 12. Relative contributions to the total potential energy in the $n = 1$, propagating wave in a water filled steel pipe, ; $T_c = R/60$. ——, Axial in-plane; - - - -, in-plane shear; · · · · , circumferential in-plane; —*—*—, axial bending; - -*- - -*- -, bending twist; · · ·*· · ·, circumferential bending.

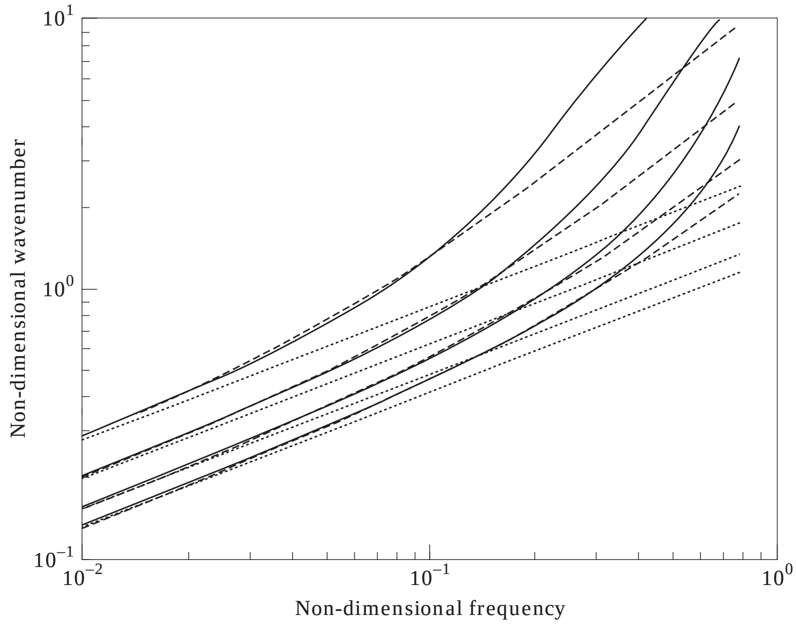


Figure 13. Wavenumbers times radius for a fluid filled steel pipe, $T_c = R/60$, $n = 1$; $c_f = 1500$ m/s. $\rho_f = 6400$, upper curves; $\rho_f = 1600$; $\rho_f = 400$; $\rho_f = 100$, lower curves. ——, Arnold and Warburton theory; - - - -, equivalent Timoshenko beam theory plus bending twist, equations (18), (19) and (29); · · · · , equivalent Euler beam theory, equation (31).

indicates that the Poisson coupling term actually reduces the total stiffness of the pipe at low frequencies.

In view of this, it is suggested that the equivalent beam approximation, equations (16), are modified to be

$$EI_n\, \partial^2\theta/\partial x^2 = GAK_n\,(\theta + \partial w/\partial x) - \rho\omega^2 I_n\,\theta,$$

$$GAK_n\left[\frac{\partial}{\partial x}\left(\theta + \frac{\partial w}{\partial x}\right) + C_n\frac{\partial^2 w}{\partial x^2}\right] = (K_w - \omega^2 M_e)w, \tag{37}$$

where $w$ is the radial displacement, $\theta$ is related to the axial displacement in equation (11), $E$ is Young's modulus, $G$ is the shear modulus, $\rho$ is the density and $A$ is the cross-sectional area of the cylinder. The parameters $I_n$, $K_n$, $K_w$, $C_n$ and $M_e$ are found in equations (14), (15), (17) and (29). The term proportional to $C_n$ is the stiffness against twist of the shell wall. At lower non-dimensional frequencies this term may be neglected. Equation (37) is then identical to the equations of motion for a Timoshenko beam on a Winkler spring foundation. At very low frequencies equation (37) is approximated by the Euler beam equations

$$EI_n\,\partial^4 w/\partial x^4 + K_w\,w - \omega^2 M_e\,w = 0, \qquad \theta = \partial w/\partial x. \tag{38}$$

The propagating wavenumbers for a water filled steel pipe with wall thickness $T_c = R/60$ was calculated by using equations (37), (38) and the routines in reference [7]. In Figure 14 are shown the relative differences between the results, showing that the beam approximations are accurate at low frequencies.

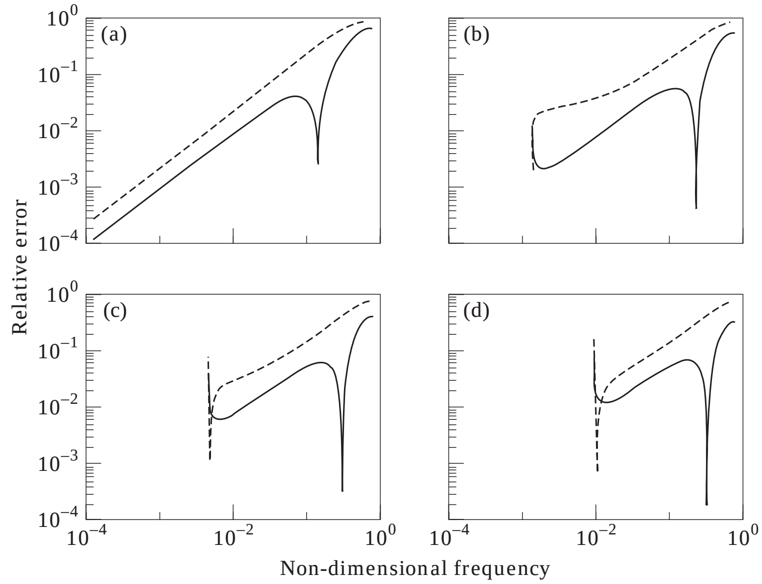


Figure 14. Relative errors in wavenumbers $T_c = R/180$. ——, Equivalent Timoshenko beam plus bending twist, equation (37); - - - -, equivalent Euler beam, equation (38). (a) $n = 1$; (b) $n = 2$; (c) $n = 3$; (d) $n = 4$.

The equivalent beam theory (16) has the strength of being derived by using three well defined assumptions, which are long axial wavelengths, frequencies well below the cut-on of higher order fluid modes, and inextensional circumferential in-plane motion. In contrast, the modifications (37) and (38) are achieved through experiments and by guessing. At higher frequencies, it is believed, the accurate rigidity is that of a shell, so the original versions of these equations should be used. At high frequencies, however, the cross-sectional bending does not largely restrain the motion, so not much harm is done if the rod rigidity is used for all frequencies, particularly as the distinction between these rigidities is, perhaps, irrelevant in most noise and vibration problems.

## 4. CONCLUSIONS

The equations of motion for straight fluid filled pipes are approximated to be similar to those for a Timoshenko beam on a Winkler foundation. Numerical experiments, and results in the literature [16, Table 2.11], reveal that the propagating waves in cylinders are as if the circumferential motion were inextensional. This is the fundamental assumption for the analysis. The derivation is also based on the assumption of long axial wavelengths, resulting in that the axial inertia of the fluid and the axial bending stiffness of the pipe wall are disregarded. The formulation is restricted to frequencies well below the cut-on of higher order fluid modes, as expressed in equation (35). For such frequencies, the compressibility of the fluid is neglected and the internal fluid loading, on the pipe, is approximated as an increase in the radial inertia.

Numerical experiments have been made, comparing the approximate theory with previously reported theory [7] by using the Helmholtz equation for the fluid and accurate thin walled cylinder theory. The free motion of fluid filled pipes is a function of non-dimensional numbers: $\nu$, $c_f/c_L$, $\rho_f/\rho$, $\beta$, $n$ and $\Omega$. By a systematic use of these non-dimensional numbers, the investigations for water filled steel pipes are thought to be complete, resulting in the criteria (33), (35) and (36) for the applicability of the simplified theory. Investigations were made varying the ratios of sound speeds and densities. These experiments justify, while not prove, the application of these criteria also for other materials.

The investigations are mainly concerned with the propagating waves in pipes. These govern the energy propagation. As will be reported, energy related properties such as modal density, input mobility and group velocity are defined in closed form by using the theory developed [17]. At restrictions in a pipe, however, the evanescent near-field waves may be needed to fulfil boundary and coupling conditions. For pipes joined with flanges, the pipe-wall is too stiff when the flexibility in the evanescent modes is neglected [22]. Similarly, at a very rigid flange almost entirely blocking the cylinder motion, energy is transmitted through the fluid. To model this energy flow, the higher order evanescent modes are needed. In pipes the near-field only extends one, or a few, pipe diameters from the restriction. When using the FEM, it may therefore be possible to use detailed models only at the restrictions, while within the pipes the equivalent beam theory presented could be applied, thus saving much calculation effort.

## ACKNOWLEDGMENTS

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